CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS OF THE QUANTUM SUPER GROUP $GL_q(3|1)$

NGUYÊN THI PHUONG DUNG, PHÙNG HÔ HAI, AND NGUYÊN HUY HUNG

Abstract. In this note, we construct all irreducible representations of the quantum general linear super group $GL_q(3|1)$ using the double Koszul complex.

1. Introduction

A quantum general linear super group is understood as a Hopf super algebra determined in terms of a Hecke symmetry R on a super vector space V of finite dimension. A representation of such a quantum group is nothing but a comodule on the corresponding Hopf super algebra.

The main invariant of a Hecke symmetry is its birank. It is shown in [7] that the category of representations of this quantum group is uniquely determined up to braided monoidal equivalence by the birank of the Hecke symmetry R, provided that the quantum parameter q is not a root of unity of order larger than 1. Therefore, the quantum general linear super group associated to a Hecke symmetry of birank (r, s) is denoted simply by $GL_q(r|s)$.

An explicit construction of irreducible representations, i.e. simple comodules over the associated Hopf super algebra, is however not known. Actually, such a construction is not known even in the classical situation of the Lie super algebras $\mathfrak{gl}(m|n)$. The difficulty lies in the so called atypical representations.

Some particular cases of lower biranks $(1|1)$ and $(2|1)$ are treated in [5, 1]. Recently, an explicit construction of irreducible representations of $\mathfrak{gl}(3|1)$ was obtained in [2] using the so called double Koszul complex. In this work, this construction will be extended to the case of quantum general linear super group $GL_q(3|1)$. To show that the representations obtained are indeed irreducible and furnish all irreducible representations we use a result of [17] on the perfect paring between $GL_q(r|s)$ and $\mathcal{U}_q(\mathfrak{gl}(r|s))$ as well as the character formula for these representations.

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2. The quantum general linear supergroup

Let V be a super vector space of finite dimension over $\Bbbk$, an algebraically closed field of characteristic zero. Fix a homogeneous basis $x_1, x_2, \ldots, x_d$ of V. We shall denote the parity of the basis element x_i by $\hat{i}$. An even operator R on $V \otimes V$ can be given by a matrix R_{ij}^{kl}:

$$R(x_i \otimes x_j) = x_k \otimes x_l R_{ij}^{kl}$$

R is an even operator implies that the matrix elements R_{kl}^{ij} are zero, except for those with $\hat{i} + \hat{j} = \hat{k} + \hat{l}$. R is called *Hecke symmetry* if the following conditions are satisfied:

i) R satisfies the Yang-Baxter equation $R_1R_2R_1 = R_2R_1R_2$, where $R_1 := R \otimes I$, $R_2 := I \otimes R$, I denotes the identity matrix of degree d.
ii) R satisfies the Hecke equation $(R-q)(R+1) = 0$ for some q which will be assumed *not to be a root of unity* of order larger than 1.
iii) There exists a matrix P_{ij}^{kl} such that $P_{jn}^{im} R_{ml}^{nk} = \delta_l^i \delta_j^k$.

Example. The following main example of Hecke symmetries was first considered by Manin [13]. Assume that the variables x_i, $i \leq r$ are even and the rest $s = d - r$ variables are odd. Define, for $1 \leq i, j, k, l \leq r + s$,

$$R^{(r|s)}{}_{ij}^{kl} := \begin{cases} q^2 & \text{if} \quad i = j = k = l, \hat{i} = 0 \\ -1 & \text{if} \quad i = j = k = l, \hat{i} = 1 \\ q^2 - 1 & \text{if} \quad k = i < j = l \\ (-1)^{\hat{i}\hat{j}} q & \text{if} \quad k = j \neq i = l \\ 0 & \text{otherwise.} \end{cases}$$

The Hecke equation for $R^{(r|s)}$ is $(x - q^2)(x + 1) = 0$. When $q = 1$, $R^{(r|s)}$ reduces to the super-permuting operator on $V \otimes V$.

Let $\{z_j^i, t_j^i | 1 \leq i, j \leq d\}$ be a set of variables, where the parities of x_j^i and t_j^i are $\hat{i} + \hat{j}$.

The super algebra E_R to be the quotient algebra of the free non-commutative algebra on the generators $\{z_j^i | 1 \leq i, j \leq d\}$, by the relations

$$(-1)^{\hat{s}(\hat{i}+\hat{p})} R_{ps}^{kl} z_i^p z_j^s = (-1)^{\hat{l}(\hat{q}+\hat{k})} z_q^k z_n^l R_{ij}^{qn}, \quad 1 \leq i, j, k, l \leq d. \tag{1}$$

Here, we use the convention of summing up over the indices that appear in both lower and upper places.

The super algebra H_R is defined to be the quotient of the free non-commutative algebra generated by $\{z_j^i, t_j^i | 1 \leq i, j \leq d\}$, by the relations

$$(-1)^{\hat{s}(\hat{i}+\hat{p})} R_{ps}^{kl} z_i^p z_j^{\mathsf{S}} = (-1)^{\hat{l}(\hat{q}+\hat{k})} z_q^k z_n^l R_{ij}^{qn}, \quad 1 \leq i, j, k, l \leq d, \tag{2}$$

$$(-1)^{\hat{j}(\hat{j}+\hat{k})} z_j^i t_k^j = (-1)^{\hat{l}(\hat{l}+\hat{i})} t_l^i z_k^l = \delta_k^i, \quad 1 \leq i, k \leq d. \tag{3}$$

The super algebra E_R is a super bialgebra with the coproduct given by

$$\Delta(z_j^i) = z_k^i \otimes z_j^k, \quad \Delta(t_j^i) = t_j^k \otimes t_k^i.$$

The super algebra H_R is a Hopf super algebra with the coproduct given by

$$\Delta(z^i_j) = z^i_k \otimes z^k_j, \quad \Delta(t^i_j) = t^k_j \otimes t^i_k,$$

and the antipode given by

$$S(z^i_j) = (-1)^{\hat{j}(\hat{i}+\hat{j})} t^i_j, \quad S(t^i_j) = (-1)^{\hat{i}(\hat{i}+\hat{j})} C^i_k z^k_l {C^{-1}}^l_j,$$

where $C^i_j := P^{il}_{jl}$. See [6] for details.

The super bialgebra E_R is called the (function algebra on) a quantum matrix super semigroup $\mathrm{M}_q(r|s)$. The Hopf super algebra H_R is called the (function algebra on) a quantum general linear group $\mathrm{GL}_q(r|s)$. When $R = R^{(r|s)}$ the associated Hopf super algebra is called the (function algebra on) standard quantum general linear super group $\mathrm{GL}_q(r|s)$. Note that $R^{(r|s)}$ has birank (r, s).

The Hecke algebra of type A, $\mathcal{H}_n = \mathcal{H}_{n,q}$ is generated by elements $T_i, 1 \le i \le n-1$, subject to the relations

$$\begin{aligned} &T_iT_j = T_jT_i, |i-j| \ge 2;\\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1};\\ &T_i^2 = (q-1)T_i + q. \end{aligned}$$

To each element w of the symmetric group $\mathfrak{S}_n$, one can associate in a canonical way an element T_w of $\mathcal{H}_n$, in particular, $T_1 = 1, T_{(i,i+1)} = T_i$. The set $\{T_w | w \in \mathfrak{S}_n\}$ forms a $\Bbbk$ basis for $\mathcal{H}_n$.

The operator R induces an action of the Hecke algebra $\mathcal{H}_n$ on the tensor powers $V^{\otimes n}$ of V, $\rho_n(T_i) = R_i := \mathrm{id}_V^{i-1} \otimes R \otimes \mathrm{id}_V^{n-i-1}$. We shall therefore use the notation $R_w := \rho(T_w)$. On the other hand, E_R coacts on V by $\delta(x_i) = x_j \otimes z^j_i$. Since E_R is a bialgebra, it coacts on $V^{\otimes n}$ by means of its multiplication. With the assumption that q is not a root of unity of order larger than 1, $\mathcal{H}_n$ is semi-simple and we have the double centralizer theorem asserting that the action and coation mentioned here are centralizers of each other in $\mathrm{End}_\Bbbk(V^{\otimes n})$ [6]. It follows that E_R-comodules are semi-simple and each simple E_R-comodule is the image of the operator induced by a primitive idempotent of $\mathcal{H}_n$ and, conversely, each primitive idempotent of $\mathcal{H}_n$ induces an E_R comodule which is either zero or simple. Since irreducible representations of $\mathcal{H}_n$ are parameterized by partitions of n, primitive idempotents of $\mathcal{H}_n$, up to conjugation, are parameterized by partitions of n, too.

For example, using the notation

$$[n] := \frac{q^n - 1}{q - 1}; \quad [n]! := [1][2]\dots[n],$$

we have the (central) primitive idempotents

$$x_n := \frac{1}{[n]!}\sum_w T_w \quad \text{and} \quad y_n := \frac{1}{[n]!} q^{n(n-1)/2} \sum_w (-q)^{-l(w)} T_w,$$

which induce the symmetrizing and anti-symmetrizing operators X_n, resp. Y_n, on $V^{\otimes n}$. Let $\mathsf{S}_n := \mathrm{Im} X_n$ and $\bigwedge_n := \mathrm{Im} Y_n$. One can show that S_n (resp. $\bigwedge_n$) is isomorphic to

the n-th homogeneous compoment of the quadratic algebra $\mathsf{S}(V)$ (resp. $\bigwedge(V)$) defined as follows:

$$\mathsf{S} \cong T(V)/(\mathrm{Im}(R-q)), \quad \text{(resp. } \textstyle\bigwedge \cong T(V)/(\mathrm{Im}(R+1))),$$

($T(V)$ denotes the tensor super algebra on V). These algebras are called the *symmetric and exterior tensor algebras* on a quantum super space.

By definition, the Poincaré series $P_{\bigwedge}(t)$ of $\bigwedge$ is $\sum_{n=0}^{\infty} \dim_{\Bbbk}(\bigwedge_n)t^n$. It is proved that this series is a rational function having only real negative roots and real positive poles [4]. Let r be the number of its roots and s be the number of its poles. Then simple E_R-comodules are parameterized by hook-partitions from $\Gamma_n^{rs} := \{\lambda \vdash n | \lambda_{r+1} \le s\}$ [6].

Simple H_R-comodules are much more complicated. The main difficulty lies in the fact that H_R-comodules are not semi-simple. In [7] it is shown that, as a braided monoidal category, the category of H_R-comodules depends only on the quantum parameter q and the birank of R. Thus the problem reduces to the case of the standard deformation $R^{(r|s)}$. In this case the problem was studied by R.B. Zhang, et.al. [15, 17], using the duality between $H_{R^{(r|s)}}$ and $\mathcal{U}_q(\mathfrak{gl}(r|s))$.

The problem of constructing all its simple comodules is still open. The aim of this work is to treat this problem in the particular case, when R has birank $(3,1)$.

3. The double Koszul complex

3.1. **The Koszul complex K.** The Koszul complex K associated to R can be defined as a collection of complexes K_a. The terms of K_a are indexed by pairs (k,l) with $k-l=a$. Denote by $\mathrm{db} : \Bbbk \to V \otimes V^*$ the map $1 \mapsto x_i \otimes \xi^i$, where (ξ^i) is the basis of V^*, dual to the basis (x_i) of V. The term $K_{k,l}$ is $\bigwedge_k \otimes {\mathsf{S}_l}^*$ and the differential $d_{k,l} : \bigwedge_k \otimes {\mathsf{S}_l}^* \to \bigwedge_{k+1} \otimes {\mathsf{S}_{l+1}}^*$ is given by:

$$d_{k,l} : \textstyle\bigwedge_k \otimes {\mathsf{S}_l}^* \longrightarrow V^{\otimes k} \otimes V^{*\otimes l} \xrightarrow{\mathrm{id}\otimes\mathrm{db}\otimes\mathrm{id}} V^{\otimes k+1} \otimes V^{*\otimes l+1} \xrightarrow{Y_{k+1}\otimes {X_{l+1}}^*} \textstyle\bigwedge_{k+1} \otimes {\mathsf{S}_{l+1}}^*,$$

where X_l, Y_k are the q-symmetrizing operators introduced in Section 2. The reader is referred to [3] for the proof that d is a differential.

Define the maps $\partial_{k,l}$ as follows:

$$\partial_{k,l} : \textstyle\bigwedge_{k+1} \otimes {\mathsf{S}_{l+1}}^* \to V^{\otimes k+1} \otimes V^{*\otimes l+1} \xrightarrow{\mathrm{id}\otimes(\mathrm{ev} R_{V,V^*})\otimes\mathrm{id}} V^{\otimes k} \otimes V^{*\otimes l} \xrightarrow{Y_k \otimes {X_l}^*} \textstyle\bigwedge_k \otimes {\mathsf{S}_l}^*,$$

where $\mathrm{ev} : V^* \otimes V \to \Bbbk$ is the evaluation map and $R_{V,V^*} : V \otimes V^* \to V^* \otimes V$ is the symmetry induced from R. In terms of the dual bases (x_i) and (ξ^j) it is given by $x_i \otimes \xi^j \mapsto \xi^k \otimes x_l P_{ik}^{jl}$, thus $\mathrm{ev}R_{V,V^*}(x_i \otimes \xi^j) = C_j^i$.

One can show [3, 7] that ∂ is also a differential and satisfies

$$q[l][k]d\partial + [l+1][k+1]\partial d = q^k([l-k]-[r-s])\mathrm{id} \tag{4}$$

on $K_{k,l}$, where (r,s) is the birank of R. Consequently, the complex K_a is exact if $a \neq s-r$. Further, it is shown that, for $a = s-r$, the complex K_a is exact everywhere, except at the term $K_{r,s}$, where it has the one dimensional homology group.

3.2. **The Koszul Complex L.** There is another Koszul complex associated to V, which was first defined by Priddy as a free resolution of the symmetric tensor algebra of V (see [12]). As in the case of the complex K, the complex L is a collection of complexes L_a. The complex L_a has (p,r)-term, with $p+r=a$, $L_{p,r} := \mathsf{S}_p \otimes \bigwedge_r$ and differential $P_{p,r} : L_{p,r} \longrightarrow L_{p-1,r+1}$ given by

$$P_{p,r} : \mathsf{S}_p \otimes \textstyle\bigwedge_r \hookrightarrow V^{\otimes p} \otimes V^{\otimes r} \xrightarrow{X_{p-1}\otimes Y_{r+1}} \mathsf{S}_{p-1} \otimes \textstyle\bigwedge_{r+1} .$$

The complexes $(L_a, P), a \geq 1$, are exact. This is shown by considering the map $Q_{p,r} : L_{p-1,r+1} \longrightarrow L_{p,r}$, given by

$$Q_{p,r} : \mathsf{S}_{p-1} \otimes \textstyle\bigwedge_{r+1} \hookrightarrow V^{\otimes p-1} \otimes V^{\otimes r+1} \xrightarrow{X_p \otimes Y_r} \mathsf{S}_p \otimes \textstyle\bigwedge_r .$$

One checks [3] that on $L_{p,r}$

$$[r][p+1]PQ + [p][r+1]QP = [p+r]\mathrm{id}. \tag{5}$$

Remark 3.1. The differentials of both complexes are morphisms of H_R-comodules.

3.3. **The double Koszul complex.** The two Koszul complexes mentioned in the previous section can be combined into a double complex called the double Koszul complex. For simplicity we shall use the dot "$\cdot$" to denote the tensor product. Fix an integer a. We arrange the Koszul complexes $K_{-a}, K_{-a-1}, K_{-a-2}, \ldots$ as follows.

$$\begin{array}{llllllll}
K_{-a} & : 0 \longrightarrow S_a{}^* & \xrightarrow{d_{0,a}} \bigwedge_1 \cdot \mathsf{S}_{a+1}{}^* & \xrightarrow{d_{1,a+1}} \bigwedge_2 \cdot \mathsf{S}_{a+2}{}^* & \xrightarrow{d_{2,a+2}} \bigwedge_3 \cdot \mathsf{S}_{a+3}{}^* & \longrightarrow \cdots \\
K_{-a-1} : & \quad 0 \longrightarrow & \mathsf{S}_{a+1}{}^* & \xrightarrow{d_{0,a+1}} \bigwedge_1 \cdot \mathsf{S}_{a+2}{}^* & \xrightarrow{d_{1,a+2}} \bigwedge_2 \cdot \mathsf{S}_{a+3}{}^* & \longrightarrow \cdots \\
K_{-a-2} : & & \quad 0 \longrightarrow & \mathsf{S}_{a+2}{}^* & \xrightarrow{d_{0,a+2}} \bigwedge_1 \cdot \mathsf{S}_{a+3}{}^* & \longrightarrow \cdots
\end{array}$$

here S_i and $\bigwedge_i$ are set to 0 if $i < 0$. To get the entries on a column into a complex we tensor each complex K_i with S_{-a-i}, i.e. the complex K_{-1-a} is tensored with S_1, the complex K_{-2-a} is tensor with S_2... Then each column can be interpreted as the complexes L_j tensored with $\mathsf{S}_{a+j}{}^*$. Thus we have the following diagram with all rows being the Koszul complexes $K_\bullet$ tensored with $\mathsf{S}_\bullet$ and columns are the Koszul complexes $L_\bullet$ tensored with $\mathsf{S}_\bullet{}^*$:

$$\begin{tikzcd}
 & 0 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathsf{S}_a{}^* \arrow[u] \arrow[r, "d"] & \bigwedge_1 \cdot \mathsf{S}_{a+1}{}^* \arrow[u] \arrow[r, "d"] & \bigwedge_2 \cdot \mathsf{S}_{a+2}{}^* \arrow[u] \arrow[r, "d"] & \bigwedge_3 \cdot \mathsf{S}_{a+3}{}^* \arrow[u] \arrow[r, "d"] & \cdots \\
 & 0 \arrow[u] \arrow[r] & \mathsf{S}_1 \cdot \mathsf{S}_{a+1}{}^* \arrow[u, "P"] \arrow[r, "d"] & \mathsf{S}_1 \cdot \bigwedge_1 \cdot \mathsf{S}_{a+2}{}^* \arrow[u, "P"] \arrow[r, "d"] & \mathsf{S}_1 \cdot \bigwedge_2 \cdot \mathsf{S}_{a+3}{}^* \arrow[u, "P"] \arrow[r, "d"] & \cdots \\
 & & 0 \arrow[u] \arrow[r] & \mathsf{S}_2 \cdot \mathsf{S}_{a+2}{}^* \arrow[u, "P"] \arrow[r, "d"] & \mathsf{S}_2 \cdot \bigwedge_1 \cdot \mathsf{S}_{a+3}{}^* \arrow[u, "P"] \arrow[r, "d"] & \cdots \\
 & & & 0 \arrow[u] & \vdots \arrow[u] &
\end{tikzcd} \tag{6}$$

A general square in diagram (6) has the form

$$\begin{tikzcd}
\mathsf{S}_i \cdot \bigwedge_k \cdot \mathsf{S}_l{}^* \arrow[r, "\mathrm{id}\otimes d"] & \mathsf{S}_i \cdot \bigwedge_{k+1} \cdot S_{l+1}{}^* \\
\mathsf{S}_{i+1} \cdot \bigwedge_{k-1} \cdot S_l{}^* \arrow[u, "P\otimes\mathrm{id}"] \arrow[r, "\mathrm{id}\otimes d"'] & \mathsf{S}_{i+1} \cdot \bigwedge_k \cdot \mathsf{S}_{l+1}{}^* \arrow[u, "P\otimes\mathrm{id}"']
\end{tikzcd} \tag{7}$$

with $l = i + k + a$. For convenient, we denote $d := \mathrm{id} \otimes d, P := P \otimes \mathrm{id}$. It is easy to show that $Pd = dP$ for all these squares. Thus (6) is a bicomplex.

We also have an exact double Koszul complex with d, P replaced by ∂, Q.

$$\begin{tikzcd}
 & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & \\
0 & \mathsf{S}_a{}^* \arrow[l, dashed, "\partial"] \arrow[d, dashed, "Q"] & \bigwedge_1 \cdot \mathsf{S}_{a+1}{}^* \arrow[l, dashed, "\partial"] \arrow[d, dashed, "Q"] & \bigwedge_2 \cdot \mathsf{S}_{a+2}{}^* \arrow[l, dashed, "\partial"] \arrow[d, dashed, "Q"] & \bigwedge_3 \cdot \mathsf{S}_{a+3}{}^* \arrow[l, dashed, "\partial"] \arrow[d, dashed, "Q"] & \cdots \arrow[l, dashed, "\partial"] \\
 & 0 & \mathsf{S}_1 \cdot \mathsf{S}_{a+1}{}^* \arrow[l, dashed, "\partial"] \arrow[d, dashed, "Q"] & \mathsf{S}_1 \cdot \bigwedge_1 \cdot \mathsf{S}_{a+2}{}^* \arrow[l, dashed, "\partial"] \arrow[d, dashed, "Q"] & \mathsf{S}_1 \cdot \bigwedge_2 \cdot \mathsf{S}_{a+3}{}^* \arrow[l, dashed, "\partial"] \arrow[d, dashed, "Q"] & \cdots \arrow[l, dashed, "\partial"] \\
 & & 0 & \mathsf{S}_2 \cdot \mathsf{S}_{a+2}{}^* \arrow[l, dashed, "\partial"] \arrow[d, dashed] & \mathsf{S}_2 \cdot \bigwedge_1 \cdot \mathsf{S}_{a+3}{}^* \arrow[l, dashed, "\partial"] \arrow[d, dashed] & \cdots \arrow[l, dashed, "\partial"] \\
 & & & 0 & \vdots &
\end{tikzcd} \tag{8}$$

From now, we assume that R has birank $(3|1)$.

We combine the two diagrams (6) and (8) into one:

$$\begin{tikzcd}
\mathsf{S}_{i-1} \cdot \mathsf{S}_{a+i-1}{}^* \arrow[r, hook, shift left, "d_{0,a+i-1}"] & \mathsf{S}_{i-1} \cdot \bigwedge_1 \cdot \mathsf{S}_{a+i}{}^* \arrow[l, dashed, two heads, shift left, "\partial_{0,a+i-1}"] \arrow[r, shift left, "d_{1,a+i}"] \arrow[d, dashed, shift left, two heads, "Q"] & \mathsf{S}_{i-1} \cdot \bigwedge_2 \cdot \mathsf{S}_{a+i+1}{}^* \arrow[l, dashed, shift left, "\partial_{1,a+i}"] \arrow[r, shift left, "d_{2,a+i+1}"] \arrow[d, dashed, shift left, "Q"] & \mathsf{S}_{i-1} \cdot \bigwedge_3 \cdot \mathsf{S}_{a+i+2}{}^* \cdots \arrow[l, dashed, shift left, "\partial_{2,a+i+1}"] \arrow[d, dashed, shift left, "Q"] \\
 & \mathsf{S}_i.\mathsf{S}_{a+i}{}^* \arrow[u, hook, shift left, "P"] \arrow[r, hook, shift left, "d_{0,a+i}"] & \mathsf{S}_i.\bigwedge_1.\mathsf{S}_{a+i+1}{}^* \arrow[u, shift left, "P"] \arrow[l, dashed, two heads, shift left, "\partial_{0,a+i}"] \arrow[r, shift left, "d_{1,a+i+1}"] \arrow[d, dashed, shift left, two heads, "Q"] & \mathsf{S}_i \cdot \bigwedge_2 \cdot \mathsf{S}_{a+i+2}{}^* \cdots \arrow[u, shift left, "P"] \arrow[l, dashed, shift left, "\partial_{1,a+i+1}"] \arrow[d, dashed, shift left, "Q"] \\
 & & \mathsf{S}_{i+1}.\mathsf{S}_{a+i+1}{}^* \arrow[u, hook, shift left, "P"] \arrow[r, hook, shift left, "d_{0,a+i+1}"] & \mathsf{S}_{i+1} \cdot \bigwedge_1 \cdot \mathsf{S}_{a+i+2}{}^* \cdots \arrow[u, shift left, "P"] \arrow[l, dashed, two heads, shift left, "\partial_{0,a+i+1}"]
\end{tikzcd} \tag{9}$$

Proposition 3.2. *Assume that the Hecke symmetry R has birank $(3, 1)$. Then the composed map $\partial PQd : \mathsf{S}_i \cdot \mathsf{S}_{a+i}{}^* \longrightarrow \mathsf{S}_i \cdot \mathsf{S}_{a+i}{}^*$ in diagram (9) is an isomorphism for all a, i with $i, a + i \geq 0$. Consequently $\mathsf{S}_i \cdot \mathsf{S}_a{}^*$ is isomorphic to a direct summand of $\mathsf{S}_{i+1} \cdot \mathsf{S}_{a+1}{}^*$. Moreover this isomorphism is an isomorphism of H_R-comodules.*

Proof. We will use induction on i to prove that the endomorphism $\partial PQd : \mathsf{S}_i{\cdot}\mathsf{S}_{a+i}{}^* \longrightarrow \mathsf{S}_i \cdot \mathsf{S}_{a+i}{}^*$ is diagonalizable with the set of eigenvalues equal to

$$A_i := \left\{ \frac{([a + 2i + 1 - j] - [-2])[j]}{[i + 1][a + i + 1]}, j = 1, 2, \dots, i + 1 \right\} \tag{10}$$

For $i = 0$, the map $PQ : \mathsf{S}_a{}^* \longrightarrow \mathsf{S}_a{}^*$ is equal to $\mathrm{id}_{\mathsf{S}_a{}^*}$. Hence

$$g = \partial PQd = \frac{[a] - [-2]}{[a+1]}\mathrm{id}.$$

Assume that the claim holds true for $i - 1$. We have

$$\begin{aligned} h := \partial PQd =& \partial[\frac{[i+1] - [2][i]QP}{[i+1]}]d = \partial d - \frac{[2][i]}{[i+1]}\partial QPd \\ =& \partial d - \frac{[2][i]}{[i+1]}Q\frac{[q([a+i-1] - [-2]) - [a+i]d\partial]}{[2][a+i+1]}P \\ =& \partial d - \frac{q[i][([a+i-1] - [-2])]}{[i+1][a+i+1]}QP + \frac{[i][a+i]}{[i+1][a+i+1]}Qd\partial P \\ =& \Big[\frac{[a+i] - [-2]}{[a+i+1]} - \frac{q[i]([a+i-1] - [-2]))}{[i+1][a+i+1]}\Big]\mathrm{id} + \frac{[i][a+i]}{[i+1][a+i+1]}Qd\partial P. \end{aligned}$$

By assumption $\partial PQd : \mathsf{S}_{i-1} \cdot \mathsf{S}_{a-1}{}^* \longrightarrow \mathsf{S}_{i-1} \cdot \mathsf{S}_{a-1}{}^*$ is diagonalizable with eigenvalues in A_{i-1},, in particular it is invertible. Thus the minimal polynomial $P(X)$ of this operator has no multiple roots. It follows that the minimal polynomial of the operator $Qd\partial P : \mathsf{S}_i \cdot \mathsf{S}_{a+i}{}^* \longrightarrow \mathsf{S}_i \cdot \mathsf{S}_{a+i}{}^*$ is just $XP(X)$. Consequently $Qd\partial P$ is diagonalizable with eigenvalues in $A_{i-1} \cup \{0\}$. Thus $\partial PQd : \mathsf{S}_i \cdot \mathsf{S}_a{}^* \longrightarrow \mathsf{S}_i \cdot \mathsf{S}_a{}^*$ is diagonalizable with the set eigenvalues in A_i .□

Consider the diagram in (6) as a exact sequence of horizontal complexes (except for the first column) and split it into short exact sequences.

$$\begin{tikzcd}
\ldots \mathrm{Ker}P_{i,k} \cdot S_{i+k+a}{}^* \arrow[r, "d'_{k,i+k+a}"] \arrow[d, dashed, shift left, "Q"] & \mathrm{Ker}P_{i,k+1} \cdot S_{i+k+a+1}{}^* \arrow[r, "d'_{k+1,i+k+a+1}"] \arrow[d, dashed, shift left, "Q"] & \mathrm{Ker}P_{i,k+2} \cdot S_{i+k+a+2}{}^* \ldots \arrow[d, dashed, shift left, "Q"] \\
\ldots \mathsf{S}_{i+1} \cdot \bigwedge_{k-1} \cdot S_{i+k+a}{}^* \arrow[r, "d_{k-1,i+k+a}"] \arrow[u, shift left, "P_{i+1,k-1}"] \arrow[d, dashed, shift left, "Q"] & \mathsf{S}_{i+1} \cdot \bigwedge_k \cdot S_{i+k+a+1}{}^* \arrow[r, "d_{k,i+k+a+1}"] \arrow[u, shift left, "P_{i+1,k}"] \arrow[d, dashed, shift left, "Q"] & \mathsf{S}_{i+1} \cdot \bigwedge_{k+1} \cdot S_{i+k+a+2}{}^* \ldots \arrow[u, shift left, "P_{i+1,k+1}"] \arrow[d, dashed, shift left, "Q"] \\
\ldots \mathrm{Ker}P_{i+1,k-1} \cdot S_{i+k+a}{}^* \arrow[r, "d'_{k-1,i+k+a}"] \arrow[u, hook, shift left, "i"] & \mathrm{Ker}P_{i+1,k} \cdot \mathsf{S}_{i+k+a+1}{}^* \arrow[r, "d'_{k,i+k+a+1}"] \arrow[u, hook, shift left, "i"] & \mathrm{Ker}P_{i+1,k+1} \cdot S_{i+k+a+2}{}^* \ldots \arrow[u, hook, shift left, "i"]
\end{tikzcd} \tag{11}$$

where $d'_{k,i+k+a}$ is the restriction of $d_{k,i+k+a}$ to $\mathrm{Ker}P_{i,k} \cdot \mathsf{S}_{i+k+a}{}^*$. Notice that $\mathrm{Ker}P_{i,j} = \mathrm{Im}P_{i+1,j-1}$ for all $i \geq 0$.

Consider the following part of (11) for $i, k \geq 1$:

$$\begin{tikzcd}
\mathsf{S}_i \cdot \bigwedge_{k+1} \cdot \mathsf{S}_{a+i+k+1}{}^* \arrow[r, shift left] \arrow[d, dashed, shift left] & \cdot\mathsf{S}_i \cdot \bigwedge_{k+2} \cdot S_{a+i+k+2}{}^* \arrow[l, dashed, shift left] \arrow[d, dashed, shift left] \\
\mathrm{Ker}P_{i,k+1} \cdot S_{a+i+k+1}{}^* \arrow[r, shift left] \arrow[u, hook, shift left] \arrow[d, dashed, shift left, "Q"] & \mathrm{Ker}P_{i,k+2} \cdot \mathsf{S}_{a+i+k+2}{}^* \arrow[l, dashed, shift left] \arrow[u, hook, shift left] \arrow[d, dashed, shift left] \\
\mathsf{S}_{i+1} \cdot \bigwedge_k \cdot S_{a+i+k+1}{}^* \arrow[r, shift left, "d"] \arrow[u, shift left, "P"] & \mathsf{S}_{i+1} \cdot \bigwedge_{k+1} \cdot \mathsf{S}_{a+i+k+2}{}^* \arrow[l, dashed, shift left, "\partial"] \arrow[u, shift left]
\end{tikzcd} \tag{12}$$

Proposition 3.3. *Assume that the Hecke symmetry R has birank $(3,1)$. Then for $i \geq 0, k \geq 1, a+i+k+1 \geq 0$ the composed map*

$$P\partial dQ : \mathrm{Ker}P_{i,k+1} \cdot \mathsf{S}_{a+i+k+1}{}^* \longrightarrow \mathrm{Ker}P_{i,k+1} \cdot \mathsf{S}_{a+i+k+1}{}^*$$

in diagram (12) is an isomorphism. Consequently $\mathrm{Ker}P_{i,k+1} \cdot \mathsf{S}_{a+i+k+1}{}^*$ is isomorphic to a direct summand of $\mathsf{S}_{i+1} \cdot \mathrm{Im} d_{k,a+i+k+1}$. Moreover the isomorphism is an isomorphism of H_R-comodules.

Proof. We assume first that $a \geq 0$, the case $a < 0$ is treated similarly but a bit more tedious. We use induction to prove that

$$P\partial dQ : \mathrm{Ker}P_{i,k+1} \cdot \mathsf{S}_{a+i+k+1}{}^* \longrightarrow \mathrm{Ker}P_{i,k+1} \cdot \mathsf{S}_{a+i+k+1}{}^*$$

is diagonalizable with eigenvalues

$$A_i := \Big\{ \frac{q^k([a+k+2i-j+2]-[-2])[j]}{[i+1][k+1]^2[a+i+k+2]}, j = 1, 2, \ldots, i+1, i+k+1 \Big\}$$

For $i = 0$, consider the following part of (12):

$$\begin{tikzcd}
\bigwedge_k \cdot \mathsf{S}_{a+k}{}^* \arrow[r, shift right, "d"'] \arrow[d, dashed, shift left, "Q"] & \bigwedge_{k+1} \cdot \mathsf{S}_{a+k+1}{}^* \arrow[l, dashed, shift right, "\partial"'] \arrow[r, shift right, "d"'] \arrow[d, dashed, shift left, "Q"] & \cdot\bigwedge_{k+2} \cdot S_{a+k+2}{}^* \arrow[l, dashed, shift right, "\partial"'] \arrow[d, dashed, shift left, "Q"] \\
\mathsf{S}_1 \cdot \bigwedge_{k-1} \cdot S_{a+k}{}^* \arrow[u, shift left, "P"] \arrow[r, shift right, "d"'] & \mathsf{S}_1 \cdot \bigwedge_k \cdot S_{a+k+1}{}^* \arrow[u, shift left, "P"] \arrow[l, dashed, shift right, "\partial"'] \arrow[r, shift right, "d"'] & \mathsf{S}_1 \cdot \bigwedge_{k+1} \cdot \mathsf{S}_{a+k+2}{}^* \arrow[u, shift left, "P"] \arrow[l, dashed, shift right, "\partial"']
\end{tikzcd}$$

The composed map $P\partial DQ : \bigwedge_{k+1} \cdot \mathsf{S}_{a+k+1}{}^* \longrightarrow \bigwedge_{k+1} \cdot \mathsf{S}_{a+k+1}{}^*$. By means of formulas (4) and (5) we have

$$\begin{aligned}
P\partial dQ &= P\frac{[q^k([a+1]-[-2]) - [k][a+k+1]d\partial]}{[k+1][a+k+2]}Q \\
&= \frac{q^k([a+1]-[-2])}{[k+1][a+k+2]}\mathrm{id} - \frac{[k][a+k+1]}{[k+1][a+k+2]}d\partial
\end{aligned}$$

Since $d\partial$ is diagonalizable with eigenvalues 0 and $\frac{[a]-[-2]}{[k+1][a+k+1]}$, $P\partial dQ$ is diagonalizable with the set of eigenvalues

$$A_0 := \Big\{ \frac{q^k[k+1]([a+1]-[-2])}{[k+1]^2[a+k+2]}, \frac{q^k([a+k+1]-[-2])}{[k+1]^2[a+k+2]} \Big\}.$$

For $i = 1$, consider diagram (12) with $i = 1$ the map $P\partial dQ : \mathrm{Ker}P_{1,k+1} \cdot \mathsf{S}_{a+k+2}{}^* \longrightarrow \mathrm{Ker}P_{1,k+1} \cdot \mathsf{S}_{a+k+2}{}^*$, we have

$$\begin{aligned}
P\partial dQ &= P\frac{[q^k([a+2]-[-2])-q[k][a+k+2]d\partial]}{[k+1][a+k+3]}Q\\
&= \frac{q^k([a+2]-[-2])}{[k+1][a+k+3]}PQ-\frac{q[k][a+k+2]}{[k+1][a+k+3]}dPQ\partial\\
&= \frac{q^k([a+2]-[-2])[k+2]}{[2][k+1]^2[a+k+3]}id-\frac{q[k][a+k+2]}{[k+1][a+k+3]}d[\frac{[k+1]-[k+1]QP}{[2][k]}]\partial\\
&= \frac{q^k([a+2]-[-2])[k+2]}{[2][k+1]^2[a+k+3]}id-\frac{q[a+k+2]}{[2][a+k+3]}d\partial+\frac{q[a+k+2]}{[2][a+k+3]}dQP\partial
\end{aligned}$$

We have $d\partial : \mathsf{S}_1\cdot\bigwedge\cdot\mathsf{S}_{a+k+2}{}^* \longrightarrow \mathsf{S}_1\cdot\bigwedge\cdot\mathsf{S}_{a+k+2}{}^*$ is diagonalizable with eigenvalues

$$\frac{q^{k+1}([a+1]-[-2])}{q[k+1][a+k+2]} \quad \text{and } 0.$$

On the other hand, we have $d\partial\cdot dQP\partial = dQP\partial\cdot d\partial$ and if $d\partial(x)=0$ than $dQP\partial(x)=0$. Therefore, the eigenvalues of $P\partial dQ : \mathrm{Ker}P_{1,k+1}\cdot\mathsf{S}_{a+k+2}{}^* \longrightarrow \mathrm{Ker}P_{1,k+1}\cdot\mathsf{S}_{a+k+2}{}^*$ are in the set

$$A_1 := \left\{\frac{q^k([a+2]-[-2])[k+2]}{[2][k+1]^2[a+k+3]},\frac{q^k([a+k+3]-[-2])}{[2][k+1]^2[a+k+3]},\frac{q^k[2]([a+k+2-[-2])}{[2][k+1]^2[a+k+3]}\right\}.$$

In general, consider the composed map

$$P\partial dQ : \mathrm{Ker}P_{i,k+1}\cdot\mathsf{S}_{a+i+k+1}{}^* \longrightarrow \mathrm{Ker}P_{i,k+1}\cdot\mathsf{S}_{a+i+k+1}{}^*$$

in diagram (12), we have

$$\begin{aligned}
P\partial dQ =& P\left[\frac{q^k([a+i+1]-[-2])-q[k][a+i+k+1]d\partial}{[k+1][a+i+k+2]}\right]Q\\
=&\frac{q^k([a+i+1]-[-2])PQ}{[k+1][a+i+k+2]}-\frac{q[k][a+i+k+1]dPQ}{[k+1][a+i+k+2]}\partial\\
=&\frac{q^k([a+i+1]-[-2])[i+k+1]\mathrm{id}}{[k+1]^2[i+1][a+i+k+2]}\\
&-\frac{q[k][a+i+k+1]d}{[k+1][a+i+k+2]}\cdot\frac{([i+k]-[i][k+1]QP)\partial}{[k][i+1]}\\
=&\frac{q^k([a+i+1]-[-2])[i+k+1]\mathrm{id}}{[k+1]^2[i+1][a+i+k+2]}-\frac{q[i+k][a+i+k+1]d\partial}{[k+1][i+1][a+i+k+2]}\\
&+\frac{q[i][a+i+k+1]dQP\partial}{[i+1][a+i+k+2]}.
\end{aligned}$$

One has $d\partial$ is diagonalizable with the set of eigenvalues

$$\left\{\frac{q^k([a+i]-[-2])}{[k+1][a+i+k+1]},0\right\}.$$

We have $d\partial \circ dQP\partial = dQP\partial \circ d\partial$ and if $d\partial(x) = 0$ then $dQP\partial(x) = 0$. By induction assumption $P\partial dQ : \mathrm{Ker}P_{i-1,k+1} \cdot \mathsf{S}_{a+i+k}{}^* \longrightarrow \mathrm{Ker}P_{i-1,k+1} \cdot \mathsf{S}_{a+i+k}{}^*$ is diagonalizable with eigenvalues in the set A_{i-1}. Thus the composed map

$$dQP\partial : \mathrm{Ker}P_{i,k+1} \cdot \mathsf{S}_{a+i+k+1}{}^* \longrightarrow \mathrm{Ker}P_{i,k+1} \cdot \mathsf{S}_{a+i+k+1}{}^*$$

is diagonalizable with the set of eigenvalues is A_i. The proof is complete.□

4. Construction of irreducible representations of $GL_q(V)$.

Let $R : V \otimes V \to V \otimes V$ be a Hecke symmetry with birank $(3|1)$. Using the double Koszul complex, we will construct in this section for each (integrable) dominant weight, i.e. a quadruple (m, n, p, t) of integers, with $m \geq n \geq p$, a comodule $I(m, n, p|t)$ of H_R. The proof that these comodules are simple and furnish all simple H_R-comodules will be given in the next section.

Recall that the complex K_2 is exact everywhere, except at the term $K_{3,1}$, where the homology is one dimensional. Denote this comodule by $I(1, 1, 1|1)$.

For a dominant weight $(m, n, p|t)$ set

$$I(m, n, p|t) := I(m - t, n - t, p - t|0) \otimes I(1, 1, 1|1)^{\otimes t}.$$

Thus one is led to construct $I(m, n, p|0)$.

First, recall from Section 2 that each partition $\lambda \in \Gamma^{3|1}$ defines a simple H_R-comodules. Denote it by M_λ. Such a partition λ has the form $(\lambda_1, \lambda_2, \lambda_3, 1^{\lambda_4})$. For a weight $(m, n, p|0)$ with $p \geq 0$ set

$$I(m, n, p|0) := M_{(m,n,p)}. \tag{13}$$

Further, for such a dominant weight with $p \geq 1$ we set

$$I(-p - 2, -n - 2, -m - 2|0) := I(m, n, p|0)^* \otimes \ I(1, 1, 1|1)^{*\otimes 3} \tag{14}$$

and for a dominant weight of type $(m, n, 0|0)$ we set

$$I(-2, -n - 1, -m - 1|0) := I(m, n, 0|0)^* \otimes \ I(1, 1, 1|1)^{*\otimes 2}. \tag{15}$$

Finally we set

$$I(-1, -1, -m|0) := I(m, 0, 0|0)^* \otimes \ I(1, 1, 1|1)^{*\otimes 1} \tag{16}$$

The reason for the choice of the weight on the left hand side above will be explained in the next section when we compute the character.

4.1. **Comodules constructed from complex** K**.** Consider complexes K_a, with $a := k - l \neq 2$.

$$K_a : \ldots \longrightarrow \textstyle\bigwedge_{k-1} \otimes \mathsf{S}_{l-1}{}^* \longrightarrow \textstyle\bigwedge_k \otimes \mathsf{S}_l{}^* \longrightarrow \textstyle\bigwedge_{k+1} \otimes \mathsf{S}_{l+1}{}^* \longrightarrow \ldots$$

By using the exactness of the complex K we will construct a class of irreducible representations of $GL_q(3|1)$. According to (4) we have

$$\textstyle\bigwedge_k . \mathsf{S}_l{}^* \cong \mathrm{Im}d_{k-1,l-1} \oplus \mathrm{Im}d_{k,l}. \tag{17}$$

For a dominant weight $(m, m, p|0)$ with $m \geq 0 > p$, set

$$I(m, m, p|0) := \mathrm{Im}d_{m+2,m-p} \otimes I(1, 1, 1|1)^{\otimes m-1}, \tag{18}$$

and

$$I(-2-p, -m-2, -m-2|0) := I(m, m, p|0)^* \otimes I(1, 1, 1|1)^{*\otimes 3}. \tag{19}$$

4.2. **Comodules constructed from the double Koszul complex.** From Proposition 3.2, for any i, a with $i, a+i \geq 0$, there exists $X_{i,a}$ such that

$$\mathsf{S}_{i+1}.\mathsf{S}_{a+i+1}{}^* = \mathsf{S}_i.\mathsf{S}_{a+i}{}^* \oplus X_{i,a}.$$

For any dominant weight $(m, -1, p|0)$ with $m \geq 0$ (and $p \leq -1$), set

$$I(m, -1, p|0) = X_{m,-m-p-1} \otimes I(1, 1, 1|1)^*. \tag{20}$$

According to Proposition 3.3, there exists comodule $Y_{i,k,a}$ such that, for i, k, a with $k \geq 1$, $i, a+i+k+1 \geq 0$,

$$\mathrm{Ker}P_{i,k+1} \otimes S_{a+i+k+1}{}^* \oplus Y_{i,k,a} \cong S_{i+1} \otimes \mathrm{Im}d_{k,a+i+k+1}.$$

For a dominant weight $(m, n, p|0)$ with $m > n \geq 0 > p$, set

$$I(m, n, p|0) = Y_{m-n-1,n+2,n-m-p-2} \otimes I(1, 1, 1|1)^{*\otimes n-1}. \tag{21}$$

For a dominant weight $(m, n, p|0)$ with $m \neq -2, n \leq -2$, we set

$$I(m, n, p|0) = I(-2-p, -2-n, -2-m)^* I(1, 1, 1|1)^{*\otimes 3}. \tag{22}$$

Thus for any integrable dominant weight $(m, n, p|0)$ we have constructed a comodule $I(m, n, p|0)$. Here is the detailed check:

(1) $m \geq n \geq 0$: $I(m, n, p|0)$ is given by (13).
(2) $m \geq n \geq 0 > p$:
  (a) $m = n$: $I(m, m, p|0)$ is given by (18);
  (b) $m > n$: $I(m, n, p|0)$ is given by (21);
(3) $m \geq 0 > n \geq p$:
  (a) $n = -1$: $I(m, -1, p|0)$ is given by (20);
  (b) $-2 \geq n$: $I(m, n, p|0)$ is given by (22);
(4) $0 > m \geq n \geq p$:
  (a) $m = n = -1$: $I(-1, -1, p|0)$ is given by (16);
  (b) $m = -1 > n$: $I(-1, n, p|0)$ is given by (22);
  (c) $m = -2$: $I(-2, n, p|0)$ is given by (15);
  (d) $-2 > m$: $I(m, n, p|0)$ is given by (22).

In the next section we shall exhibit the simplicity of these comodules by reducing it to the case of the standard Hecke symmetry $R^{(r|s)}$ and using the formal character.

5. Simplicity and completeness

In this section we shall prove the simplicity of the comodules constructed in the previous section and that they furnish all simple comodules of H_R. Our method is to use the representation theory of the quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{gl}(3|1))$. According to [7, Thm 4.3] there is a monoidal equivalence between the category of comodules over H_R and the category of comodules over $H_{R^{(3|1)}}$. Thus the problem is reduced to the case $R = R^{(3|1)}$. In this case, there is a duality between $H_{R^{(3|1)}}$ and $\mathcal{U}_q(\mathfrak{gl}(3|1))$, [17, Thm 3.5], which shows that there is an equivalence between the category of comodules over $H_{R^{(3|1)}}$ and finite dimensional integrable representations of $\mathcal{U}_q(\mathfrak{gl}(3|1))$. Notice that irreducible representations of $\mathcal{U}_q(\mathfrak{gl}(n|1))$ can be obtained by other methods, see e.g. [14, 10]. But these methods are not compatible with the braided monoidal equivalence mentioned here. This is the reason why we want to give a construction based merely on the braiding (given by R) and the two maps ev and db.

For finite dimensional representations of $\mathcal{U}_q(\mathfrak{gl}(3|1))$ the weight decomposition is obtained in the same way as for the classical case of $\mathfrak{gl}(3|1)$, whence the character is defined and does not depend on the parameter q (as long as q is not a root of unity).

The character of $H_{R^{(3|1)}}$-comodules can be defined directly. Consider the the quotient Hopf super algebra of this algebra by setting $z^i_j = 0$ for all $i \neq j$. This quotient is just the algebra of Lorenz polynomials $\Bbbk[z_i^{i\pm 1}]$. Any $H_{R^{(3|1)}}$-comodule restricts to a comodule over $\Bbbk[z_i^{i\pm 1}]$, which is semi-simple, yielding the weight decomposition, which is independent of the quantum parameter. Assume that M is a comodules over $H_{R^{(3|1)}}$, consider it as a comodule over $\Bbbk[z_i^{i\pm 1}]$ we obtain the decomposition

$$M \cong \bigoplus_{\lambda} M_\lambda,$$

where λ run over the set of $\mathbb{Z}$-linear mappings from the free abelian group generated by z_i^i to $\mathbb{Z}$, i.e. the set of integrable weights. The character of M_λ is defined to be

$$\mathrm{ch}(M_\lambda) := \sum \dim_{\Bbbk}(M_\lambda) e^{\lambda}.$$

If follows immediately from the definition that the character is additive with respect to short exact sequences and multiplicative with respect to the tensor product. The fact that this definition agrees with the above definition follows from the explicit duality between $H_{R^{(3|1)}}$ and $\mathcal{U}_q(\mathfrak{gl}(3|1))$.

Now to finish the proof that all comodules of H_R constructed in the previous section are simple and furnish all H_R-comodules, it suffices to verify the following lemma and to compute explicitly the character of these comodules.

Lemma 5.1. *Let V be a representation of $\mathcal{U}_q(\mathfrak{gl}(3|1))$ with the character equal to the character of the simple highest weight representation $V(\lambda)$. Then V is isomorphic to $V(\lambda)$.*

Proof. Consider V and $V(\lambda)$ as representations of the Hopf subalgebra $\mathcal{U}_q(\mathfrak{gl}(3) \oplus \mathfrak{gl}(1))$. Since they have the same character, they are isomorphic. In particular, as $\mathcal{U}_q(\mathfrak{gl}(3) \oplus \mathfrak{gl}(1))$-representations, V contains a direct summand with highest weight λ, say $S(\lambda)$.

According to [15], $V(\lambda)$ is obtained from $S(\lambda)$ by induction. More precisely, $V(\lambda)$ is the quotient of the Kac representation $\overline{V}(\lambda)$ by its maximal sub-representation. The representation $\overline{V}(\lambda)$ is defined as follows. One first extend (in a trivial way) the action of $\mathcal{U}_q(\mathfrak{gl}(3) \oplus \mathfrak{gl}(1))$ to the action of an intermediate algebra and then induce this action to the whole algebra $\mathcal{U}_q(\mathfrak{gl}(3|1))$.

It follows by adjoin property that there is a non-zero map

$$\overline{V}(\lambda) \to V.$$

Hence $V(\lambda)$ is a sub-quotient V. But they have the same character, in particular, same (total) dimension, hence are isomorphic. □

Lemma 5.2. *The character of the representation $I(\lambda)$ constructed in the previous section is equal to the character of the highest weight irreducible representation $V(\lambda)$ of $\mathcal{U}_q(\mathfrak{gl}(3|1))$.*

Proof. The character of $V(\lambda)$ does not depend on q, hence can be computed by classical formula, for instance it is given explicitly in [2]. On the other hand, the character of $I(\lambda)$ can be computed directly from their construction and the compatibility of the character with exact sequences and tensor product. First, setting

$$x_1 = e^{(1,0,0|0)}, x_2 = e^{(0,1,0|0)}, x_3 = e^{(0,0,1|0)}, y = e^{(0,0,0|1)},$$

we have

$$\mathrm{ch}(I(1,0,0|0)) = \chi(V) = x_1 + x_2 + x_3 - y.$$

Using [11, Example I.3.22(4)] we have, for $m \geq n \geq p \geq 1$,

$$\mathrm{ch}(I(m,n,p|0)) = (x_1x_2x_3)^{p-1}(x_1+y)(x_2+y)(x_3+y)S(m-p,n-p,0)$$

where $S(m,n,p)$ is the Schur function on the variables x_1, x_2, x_3, associated to partition (m,n,p). Further, we have

$$\mathrm{ch}(I(m,n,0|0) = \frac{(x_1+y)(x_2+y)(x_3+y)}{(x_1-x_2)(x_2-x_3)(x_1-x_3)} \times \\ \left(\frac{x_2^{m+1}x_3^n - x_2^n x_3^{m+1}}{x_1+y} + \frac{x_3^{m+1}x_1^n - x_3^n x_1^{m+1}}{x_2+y} + \frac{x_1^{m+1}x_2^n - x_1^n x_2^{m+1}}{x_3+y}\right),$$

$$\mathrm{ch}(I(m,0,0|0) = \frac{(x_1+y)(x_2+y)(x_3+y)}{(x_1-x_2)(x_2-x_3)(x_1-x_3)} \times \\ \left(\frac{x_2^{m+1} - x_3^{m+1}}{x_1+y} + \frac{x_3^{m+1} - x_1^{m+1}}{x_2+y} + \frac{x_1^{m+1} - x_2^{m+1}}{x_3+y}\right).$$

Since $I(1,1,1|1)$ gives the quantum super determinant, we have

$$\mathrm{ch}(I(1,1,1|1) = x_1x_2x_3y^{-1}.$$

Using induction we obtain, for $k-l\neq 2$, $k\geq 2$,

$$\mathrm{ch}(\mathrm{Im}d_{k,l})=\frac{(x_1+y)(x_2+y)(x_3+y)y^{k-3}}{(x_1x_2x_3)^l}S(l,l,0).$$

Hence we have, according to (18), for $m\geq 0>p$,

$$\mathrm{ch}(I(m,m,p|0)=(x_1+y)(x_2+y)(x_3+y)(x_1x_2x_3)^{p-1}S(m-p,m-p,0).$$

Next, we have, for $i,a\geq 0$,

$$\mathrm{ch}(X_{i,a})=\frac{(x_1+y)(x_2+y)(x_3+y)}{(x_1-x_2)(x_2-x_3)(x_1-x_3)y}\left(\frac{x_1(x_2^{-a-i-1}x_3^{i+2}-x_2^{i+2}x_3^{-a-i-1})}{x_1+y}+\right.$$
$$\left.+\frac{x_2(x_3^{-a-i-1}x_1^{i+2}-x_3^{i+2}x_1^{-a-i-1})}{x_2+y}+\frac{x_3(x_1^{-a-i-1}x_2^{i+2}-x_1^{i+2}x_2^{-a-i-1})}{x_3+y}\right)$$

That is, $X_{i,a}$ has the same character as the comodule $V(i+1,0,-a-i|1)$.

Finally, we have, for $i\geq 0,k\geq 2,a+i+k\geq 0$,

$$\mathrm{ch}(Y_{i,k,a})=\frac{(x_1+y)(x_2+y)(x_3+y)y^{k-3}}{(x_1x_2x_3)^{a+i+k+1}}S(a+2i+k+2,a+i+k+1,0).$$

That is $Y_{i,k,a}$ has the same character as $V(i+2,1,-a-i-k|3-k)$. This formula for the case $a+i+3\neq 0$ follows from the character formula for $d_{k,l}$ given above.

For the case $a+i+3=0$, the comodule $\mathrm{Im}d_{k,k-2}$ is not simple, its character can be computed by using the complex K_2. Indeed, have $\mathrm{Im}d_{2,0}=\bigwedge_2$. Using induction and the fact that the homology of K_2 is concentrated at the term $(3,1)$ and is $I(1,1,1|1)$ one can show that $\mathrm{Im}d_{k,k-2}$ has a decomposition series consisting of $I(1,1,2-k|2-k)$ and $I(1,1,3-k|3-k)$.

By the formulas given above one can easily check that for any dominant weight $(m,n,p|0)$

$$I(m,n,p|0)\cong V(m,n,p|0).$$

This finishes the proof. $\square$

The following theorem is a direct consequent of the two lemmas above.

Theorem 5.3. *The comodules $I(\lambda)$ constructed in the previous section are simple and furnish all simple comodules of the Hopf super algebra H_R.*

Remark 5.4. The first named author has constructed in [2] a full list of irreducible representations of the super group $GL(3|1)$. There is unfortunately several misprints in that work that makes the list in fact incomplete. The description here fulfills this gap.

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(Dung) Border Academy, Son Tay, Hanoi, Vietnam

E-mail address: phuongdung72@yahoo.com

(Hai) Institute of Mathematics, Hanoi, Vietnam

E-mail address: phung@math.ac.vn

(Hung) Hanoi Pedagogical University II, Vinhphuc, Vietnam

E-mail address: hungp1999@yahoo.com